π DIVISION and ADDITION

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The number π

In *AMT Volume 64 Number 1* we saw that the number $\pi \approx 3.14159$ is defined to be the ratio C/d of the circumference C to the diameter d of any given circle. In particular, π measures the circumference of a circle of diameter $d = 1$. Historically, the Greek mathematician Archimedes found good approximations for π by inscribing and circumscribing many-sided polygons about this circle, and calculating their perimeters.

Since π stands for an infinite decimal, for practical purposes it is useful to find fractions which have a value close to π. We look at this first. Later we look at one of several surprising appearances of π, where a circle is nowhere in sight!

Long division

Class investigation

1. (a) Have a look at the equation $x^2 - 2x - 1 = 0$. You can probably solve this. Do it.

   (b) Did you get $x = \sqrt{2} + 1$? Try putting this value of x back in the equation. Does it satisfy it? Notice that $\sqrt{2} + 1 \approx 2.414$.

   (c) Now try this. Noting that $x = 0$ is not a solution of the equation, divide the equation through by x, giving
   $$x = 2 + \frac{1}{x}$$

   (d) Now try replacing the x on the right hand side by the expression
   $$x = 2 + \frac{1}{x}$$
   for the x on the left hand side. What do you get?

   (e) You should have obtained
   $$x = 2 + \cfrac{1}{2 + \cfrac{1}{x}}$$

   (f) Now that annoying x is still on the right hand side, so why not replace it again — and again! You should now get:
   $$x = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{x}}}}$$

This is all a bit strange; but notice that this answer does indeed look like "2 and a bit" — much like $\sqrt{2} + 1$. What if we just ignore the $1/x$ at the end?

2. Try adding three more rows to the following table:

Expression	Drop off $\frac{1}{x}$	Fraction	Decimal
$2 + \frac{1}{x}$	2	2	2.000
$2 + \cfrac{1}{2 + \cfrac{1}{x}}$	$2 + \frac{1}{2}$	$\frac{5}{2}$	2.500
$2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{x}}}$	$2 + \cfrac{1}{2 + \cfrac{1}{2}}$	$2\frac{2}{5}$	2.400

You will notice that the decimals in the final column appear to approach more and more closely our expected value $x = \sqrt{2} + 1 \approx 2.414$.

In general, an expression of the form

$$x = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \dots}}}$$

is called a *simple continued fraction*. The expressions

$$a_1,\ a_1 + \frac{1}{a_2},\ a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3}},\ \dots$$

are called *convergents* of the continued fraction.

Now this is all very interesting, but what does it have to do with π? Well, we can construct a continued fraction for $\pi \approx 3.14159\ldots$ (or any other number) in the following way:

- Break off the integer part (3) to get $3 + 0.14159\ldots$
- Invert the decimal part, obtaining
$$3 + \frac{1}{7.06251\ldots}$$
- Break off the integer part (7) to get $7 + 0.06251\ldots$ and continue in this way.

If you persevere, you will obtain:

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \dots}}}}$$

3. Find the first four convergents of the simple continued fraction for π. Express each as a decimal, and compare it with the decimal expansion for π.

The convergents of the simple continued fraction for a number x have a remarkable property:

> If p/q is such a convergent, then of all fractions with denominators not exceeding q, p/q is closest to x.

Perhaps you can now see why $22/7$ is so often used as an approximate value for π.

For further investigation

There is some appeal in looking for continued fractions which have a simple form.

1. Find the continued fraction for $2 + \sqrt{5}$. This is a solution of the equation $x^2 - 4x - 1 = 0$.

2. Find the continued fraction for $(\sqrt{5} + 1)$. This is a solution of the equation $x^2 - x - 1 = 0$. Find the first few convergents of this continued fraction. They should look familiar!

Long addition

> "Can you do Addition?" the White Queen asked. "What's one and one and one and one and one and one and one and one and one and one?"
> "I don't know," said Alice. "I lost count."
> "She can't do Addition," the Red Queen interrupted.
>
> – Lewis Carroll

We can think of a continued fraction as a sort of infinite long division. We make sense of it (or evaluate it) by biting off finite chunks. We can sometimes do the same thing with long addition.

Class investigation

1. Fill in the missing entries in the following table.

Sum expression	Total
$S_1 = 1$	1
$S_2 = 1 + \frac{1}{2^2}$	1.25
$S_3 = 1 + \frac{1}{2^2} + \frac{1}{3^2}$	1.361111…
	1.423611…

The infinite sum

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}+\ldots$$

is an example of an *infinite series*. The finite expressions S_1, S_2, S_3, ... listed in the left column of the table are called *partial sums*.

2. For a series of positive terms the partial sums become larger and larger the further we go. One of two things may happen.

   (a) Consider the series 1 + 1 + 1 + 1 + ... What happens to the partial sums here?

   (b) In the series in the table, the number 1.646934... turns out to be rather special. Can you suggest why? You might need to add some extra rows to the table.

In case (b) the partial sums approach more and more closely from below the number 1.646934... It seems sensible to call this value the *sum* of the series. We say that the partial sums *converge to the sum S*.

Now, once again, what does this series have to do with π? In fact,

$$1.646934\ldots=\frac{\pi^2}{6}$$

That is, the sum of the series

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}+\ldots \text{ is } \frac{\pi^2}{6}$$

This is a real surprise, because there is not a circle in sight!

For further investigation

1. The series

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}+\ldots$$

   is very important in number theory. Its sum is usually called $\zeta(2)$ or zeta (2). The symbol zeta is just the last letter in the Greek alphabet; the 2 denotes the square power in the series. What would $\zeta(4)$ stand for? It is known that $\zeta(4) = \pi^4/90$. Check this out by calculating the first few partial sums.

2. Any work on $\zeta(2)$ quickly gets complicated! For anyone with some knowledge of integration, a nice proof that $\zeta(2)$ has the value $\pi^2/6$ is available on the website www.mathreference.com/lc-z,zeta2.html. Essentially the proof introduces the circular functions sine and cosine, and this is where the number π appears.

3. An integer is said to be square-free if it is not divisible by any square; thus 26 is square free but 18 is not. If S denotes the number of square-free numbers less than N, then it is known that S/N approaches $1/\zeta(2)$. This makes an interesting hands-on exercise.

Bibliography

Scott, P. R. (1974). *Discovering the Mysterious Numbers.* Cheshire.

For a good introduction to continued fractions, see http://en.wikipedia.org/wiki/Continued_fraction

For a good introduction to infinite series see http://en.wikipedia.org/wiki/Series_(mathematics)

From Helen Prochazka's
Scrapbook

I was in college before I realised that mathematics is not cold. It has a warm temperature. Maths has a pulse. It has a heartbeat.
Bill Cosby, comedian (1991)

Maths is like a puzzle. You have something exciting you want to do. You use the rules of mathematics to find the answer. In school the answer is an answer that everybody got because you are learning. You only solve problems that were solved before. In careers and science and engineering the answer is a discovery because no-one got that answer before.
Amar Bosc, CEO Bose Corporation (1991)